The Structure of Singularity in Gravitational Collapse

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We will describe here the structure of singularity forming in gravitational collapse of spherically symmetric inhomogeneous dust. Such a collapse is described by the Tolman-Bondi-Lemaître metric. The main new result here relates, in a general way, the formation of black holes and naked shell-focusing singularities resulting as the final fate of such a collapse to the generic form of regular initial data. Such a data is characterized in terms of the density and velocity profiles of the matter, specified on an initial time slice from which the collapse commences. We show that given any generic density profile at the initial time slice, there exists a corresponding velocity profile which gives rise to a strong curvature naked singularity. This establishes that strong naked singularities arise for a generic density profile. We also establish here that similar results hold for black hole formation as well. Keeping the model to be spherically symmetric we also consider more general form of matter fields, i.e. equation of state $p = k\rho$. We will analyse here the nature of non-central singularity forming due to collapse of spherically symmetric perfect fluid subject to weak energy condition.

1 Introduction

The gravitational collapse of a spherically symmetric homogeneous dust can result into a black hole in the space-time[1], characterized by the presence of an event horizon, and the space-time singularity at the center, which is covered by the horizon. This model provides the basic motivation for the black hole physics, and the cosmic censorship conjecture [2], which states that even when the assumptions contained in the above case are relaxed, in the form of either perturbations in the symmetry or form of matter etc., the outcome would still be a black hole in generic situations. It is clear that the assumption of homogeneity is only an idealization, and realistic density profiles, for massive objects such as stars will have inhomogeneous density distribution , peaked typically at the center of the object. In section 2 we analyse inhomogeneous dust models, called Tolman-Bondi-Lemaître (**TBL**)[3] models.

The use of a pressureless fluid in all these models is an important assumption. Analysing the Einstein equations for a more general equation of state, subject to suitable energy conditions, is astrophysically more important. Using analytical methods, similar to those developed by Podurets [4], we demonstrate the role of pressure in determining whether a naked singularity or a black hole will form in the collapse.

2 Tolman-Bondi-Lemaître (TBL) models

The TBL space-times are spherically symmetric manifolds ($\mathcal{M}$,g), with metric of the form,

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2 d\Omega^2, \tag{1}$$

and energy-momentum tensor of the form of a perfect fluid with equation of state p=0, given by

$$T^{ij} = \epsilon\, \delta^i_t \delta^j_t. \tag{2}$$

Here ϵ and R are functions of r and t, and $d\Omega^2$ is the metric on the 2-sphere. The Einstein equations become

$$\dot{R}^2 = \frac{F(r)}{R} + f(r), \quad \epsilon(r,t) = \frac{F'(r)}{R^2 R'}, \tag{3}$$

where the dot and prime signify partial derivatives with respect to t and r respectively. It is seen that the energy density blows up either at the "shell-focusing singularity" $R = 0$, or when $R' = 0$ which corresponds to a "shell-crossing singularity" in the space-time. Equation (3) leads to the interpretation of $F(r)$ and $f(r)$ as mass and energy function respectively. The model is said to be bound, unbound, or marginally bound if $f(r)$ is less than zero, greater than zero, or equal to zero.
The integrated form of equation (3) is given by

$$t - t_s(r) = -\frac{R^{3/2} G(-fR/F)}{\sqrt{F}} \tag{4}$$

where $G(y)$ is a real, positive and smooth function which is bounded, monotonically increasing and strictly convex. $t_s(r)$ is a constant of integration, fixed by the choice of scaling on the initial surface i.e. $R(0,r) = r$. The time $t = t_s(r)$ corresponds to the value $R = 0$ where the area of the matter shell at a constant value of the coordinate r vanishes, which corresponds to the physical space-time singularity. Thus the range of coordinates is given by

$$0 \leq r < r_c, \quad -\infty \leq t < t_s(r), \tag{5}$$

where $r = r_c$ denotes the boundary of the dust cloud where the solution is matched to the exterior Schwarzchild solution.
It can be shown that[5] for a generic expandable density and velocity profile, on the initial regular hypersurface, of the form

$$\rho(r) = \sum_{n=0}^{\infty} \rho_n r^n, \ \ i.e.\ \ F(r) = \sum_{n=0}^{\infty} F_n r^{n+3}, \ \ and\ \ f(r) = \sum_{n=2}^{\infty} f_n r^n. \tag{6}$$

the existence of locally naked singularity (black hole) is related to existence (absence) of a real positive root to the quartic equation

$$(\alpha - 1)x^4 + \sqrt{\Lambda_0}x^3 - \Theta_0 x + \sqrt{\Lambda_0}\Theta_0 = 0 \tag{7}$$

where $x = \sqrt{(R/r^\alpha)}$, $\Lambda = \frac{F}{r^\alpha}$ and $\Theta \equiv \frac{t_s'\sqrt{\Lambda}}{r^{\alpha-1}}$. Subscript "0" denotes values of the functions specified on initial regular hypersurface at $r = 0$. Alpha is a free parameter introduced to analyse the characteristic curves and can be uniquely fixed for a given initial data. Also it is shown[5] that both naked singularity and black hole can develop from generic form of initial data.

3 Perfect fluid models

We study here the formation of singularities in the perfect fluid space-time, equation of state $p = k\rho$, subject to weak energy condition. The metric in comoving coordinates is of the form

$$ds^2 = (\rho^{\frac{-2k}{1+k}})dt^2 - \frac{(\rho^{\frac{-2}{1+k}})}{R^4}dr^2 - R^2 d\Omega^2 \tag{8}$$

where $\rho(r,t)$ denotes energy density and $R(r,t)$ is area coordinate. Einstein's equations assumes the form

$$\begin{aligned} m &= \frac{R}{2}(1 + \rho^{\frac{2k}{1+k}}(\dot{R})^2 - R^4(\rho^{\frac{2}{1+k}})(R')^2) \\ \dot{m} &= -4\pi p_r R^2 \dot{R} \\ m' &= 4\pi\rho R^2 R' \end{aligned} \tag{9}$$

Weak energy condition i.e. $\rho \geq 0$; $\rho + p_r \geq 0$ and $\rho + p_\theta \geq 0$, implies $k \geq -1$. This allows for the possibility of negative pressure in extreme conditions during gravitational collapse. $\dot{m}$ equation above indicates that for collapse models ($\dot{R} < 0$) positivity of energy density implies that negative values of k can make "m" vanish in the limit of approach to singularity. It can be shown that[6] we can have a approximate solution near singularity and for non-central points on the singularity curve mass function "m" necessarily vanishes, as we approach singularity, for $-1 \leq k \leq -1/3$. Also the singularity is always naked in this range independent of rest of the initial data.

This indicates the possibility that the earlier results for the spherical gravitational collapse for dust equation of state might generalize to the case of perfect-fluid space-times.

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References

1. J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

2. R. Penrose, Riv. del. Nuovo Cim. **1**, 252 (1969).
3. R.C. Tolman, Proc. Natl. Acad. Sci. USA **20**, 410 (1934); H. Bondi,Mon. Not. Astron. Soc. **107**, 343 (1947).
4. M. A. Podurets, Sov. Phys. Doklady **11**, 275 (1966).
5. S. Jhingan and P. S. Joshi gr-qc/970101.
6. F. I. Cooperstock, *et al.* Class. Quant. Grav. **14**, 2195 (1997).